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Fractional Integral Curves of Some Fractional Differential Equations

Chii-Huei Yu

School of Mathematics and Statistics,

Zhaoqing University, Guangdong, China

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Abstract: **In this paper, based on Jumarie's modification of Riemann-Liouville (R-L) fractional calculus, we find the fractional integral curves of some fractional differential equations. A new multiplication of fractional analytic functions and product rule for fractional derivatives play important roles in this article. In fact, our results are generalizations of the results in ordinary differential equations.**

Keywords: **Jumarie's modification of R-L fractional calculus, fractional integral curves, fractional differential equations, new multiplication, fractional analytic functions, product rule.**

I. INTRODUCTION

Fractional calculus is a field of mathematical analysis. It studies and applies integrals and derivatives of any order. Fractional calculus originated in 1695, almost at the same time as classical calculus. In the second half of the 20th century, a large number of studies on fractional calculus were published in engineering literature. In fact, the latest development of fractional calculus is widely used in differential and integral equations, physics, mechanics, control theory, economics, viscoelasticity, biology, electrical engineering, and other fields [1-10].

However, the definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [11-14].

Based on Jumarie's modified R-L fractional calculus, we study the fractional integral curves of two kinds of fractional differential equations. A new multiplication of fractional analytic functions and product rule for fractional derivatives play important roles in this paper. In fact, our results are generalizations of these results in ordinary differential equations.

II. DEFINITIONS AND PROPERTIES

First, we introduce the fractional calculus used in this paper.

Definition 2.1 ([15])**:** Assume that $0 < \alpha \leq 1$, and x_0 is a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \tag{1}$$

And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function. Moreover, we define $\left({}_{x_0}D_x^{\alpha}\right)^n[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, and it is called the n-th order α-fractional derivative of $f(x)$, where n is any positive integer.

In the following, some properties of Jumarie's fractional derivative are proposed.

Proposition 2.2 ([16])**:** *Let* α, β, x_0, c *be real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[c] = 0. \tag{4}$$

Next, the definition of fractional analytic function is introduced.

Definition 2.3([17])**:** Let x, x_0, and a_k be real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on$[a,b]$.

Next, we introduce a new multiplication of fractional analytic function.

Definition 2.4 ([18])**:** If $0 < \alpha \leq 1$, and x_0 is a real number. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{6}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \end{aligned} \tag{8}$$

Definition 2.5 ([19])**:** Suppose that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes k}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha(g_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha(f_\alpha(x^\alpha)) = \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(f_\alpha(x^\alpha)\right)^{\otimes k}. \tag{12}$$

Definition 2.6 ([19])**:** Let $0 < \alpha \leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

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$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{13}$$

Then these two fractional analytic functions are called inverse to each other.

Definition 2.7 ([19])**:** Let $0 < \alpha \le 1$, and x be a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \tag{14}$$

The α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

In the following, the arbitrary power of fractional analytic function is defined.

Definition 2.8 ([18])**:** Suppose that $0 < \alpha \le 1$ and r is any real number. The r-th power of the α-fractional analytic function $f_\alpha(x^\alpha)$ is defined by

$$[f_\alpha(x^\alpha)]^{\otimes r} = E_\alpha\left(rLn_\alpha(f_\alpha(x^\alpha))\right). \tag{15}$$

Theorem 2.9 (product rule for fractional derivatives) ([20])**:** *Let $0 < \alpha \le 1$, and $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ be α-fractional analytic at $x = x_0$, then*

$$\left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha)] = \left({}_{x_0}D_x^\alpha\right)[f_\alpha(x^\alpha)] \otimes g_\alpha(x^\alpha) + f_\alpha(x^\alpha) \otimes \left({}_{x_0}D_x^\alpha\right)[g_\alpha(x^\alpha)]. \tag{16}$$

III. MAIN RESULTS

In this section, we introduce two major results in this paper.

Theorem 3.1: *If $0 < \alpha \le 1$, then the first order α-fractional differential equation*

$$\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] = y_\alpha(x^\alpha) + \left(\left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2} + \left(y_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes \frac{1}{2}}. \tag{17}$$

has the α-fractional integral curves

$$y_\alpha(x^\alpha) = \frac{1}{2c} \cdot \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes 2} - \frac{c}{2}. \tag{18}$$

Where c is any real number, $c \ne 0$.

Proof Since

$$\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)] = \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes -1} \otimes y_\alpha(x^\alpha) + \left(1 + \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes -2} \otimes \left(y_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes \frac{1}{2}}. \tag{19}$$

Let $u_\alpha(x^\alpha) = \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes -1} \otimes y_\alpha(x^\alpha)$, it follows that $y_\alpha(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes u_\alpha(x^\alpha)$, and hence

$$\left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes u_\alpha(x^\alpha)\right] = u_\alpha(x^\alpha) + \left(1 + \left(u_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes \frac{1}{2}}. \tag{20}$$

By product rule for fractional derivatives, we have

$$u_\alpha(x^\alpha) + \frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \left({}_0D_x^\alpha\right)[u_\alpha(x^\alpha)] = u_\alpha(x^\alpha) + \left(1 + \left(u_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes \frac{1}{2}}. \tag{21}$$

Therefore,

$$\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes \left({}_0D_x^\alpha\right)[u_\alpha(x^\alpha)] = \left(1 + \left(u_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes \frac{1}{2}}. \tag{22}$$

Thus,

$$\left({}_0D_x^\alpha\right)[u_\alpha(x^\alpha)] \otimes \left(1 + \left(u_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes -\frac{1}{2}} = \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes -1}. \tag{23}$$

So,

$$\left({}_0I_x^\alpha\right)\left[\left({}_0D_x^\alpha\right)[u_\alpha(x^\alpha)]\otimes\left(1+\left(u_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes -\frac{1}{2}}\right]=\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right]+C. \quad (24)$$

We have

$$Ln_\alpha\left(u_\alpha(x^\alpha)+\left(1+\left(u_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}\right)=Ln_\alpha(x^\alpha)-Ln_\alpha(c). \quad (25)$$

That is,

$$u_\alpha(x^\alpha)+\left(1+\left(u_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}=\frac{1}{c}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha. \quad (26)$$

And hence,

$$\left(\frac{1}{c}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha-u_\alpha(x^\alpha)\right)^{\otimes 2}=1+\left(u_\alpha(x^\alpha)\right)^{\otimes 2}. \quad (27)$$

Thus,

$$\frac{1}{c^2}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}-\frac{2}{c}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\otimes u_\alpha(x^\alpha)=1. \quad (28)$$

Therefore,

$$\frac{1}{c^2}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}-\frac{2}{c}\cdot y_\alpha(x^\alpha)=1. \quad (29)$$

Finally, we obtain

$$y_\alpha(x^\alpha)=\frac{1}{2c}\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}-\frac{c}{2}.$$

Theorem 3.2: *Let $0<\alpha\leq 1$, a be a real number, $a\neq 0$, then the initial-value problem of second order α-fractional differential equation*

$$\left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)]=\frac{1}{a}\cdot\left(1+\left(\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)]\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}, \quad (30)$$

$$y_\alpha(0)=a,\ \left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)](0)=0, \quad (31)$$

has the α-fractional integral curve

$$y_\alpha(x^\alpha)=\frac{a}{2}\cdot\left(E_\alpha\left(\frac{1}{a}x^\alpha\right)+E_\alpha\left(-\frac{1}{a}x^\alpha\right)\right), \quad (32)$$

which is called α-fractional catenary.

Proof Let $\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)]=p_\alpha(x^\alpha)$, then $\left({}_0D_x^\alpha\right)^2[y_\alpha(x^\alpha)]=\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]$. Thus,

$$\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]=\frac{1}{a}\cdot\left(1+\left(p_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}. \quad (33)$$

It follows that

$$\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes\left(1+\left(p_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes -\frac{1}{2}}=\frac{1}{a}. \quad (34)$$

And hence,

$$\left({}_0I_x^\alpha\right)\left[\left({}_0D_x^\alpha\right)[p_\alpha(x^\alpha)]\otimes\left(1+\left(p_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes -\frac{1}{2}}\right]=\left({}_0I_x^\alpha\right)\left[\frac{1}{a}\right]. \quad (35)$$

Therefore,

$$Ln_\alpha\left(p_\alpha(x^\alpha)+\left(1+\left(p_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}\right)=\frac{1}{a}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+C. \tag{36}$$

Since $\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)](0)=0$, it follows that $p_\alpha(0)=0$, and hence $C=0$. So,

$$Ln_\alpha\left(p_\alpha(x^\alpha)+\left(1+\left(p_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}\right)=\frac{1}{a}\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha. \tag{37}$$

Thus,

$$p_\alpha(x^\alpha)+\left(1+\left(p_\alpha(x^\alpha)\right)^{\otimes 2}\right)^{\otimes\frac{1}{2}}=E_\alpha\left(\frac{1}{a}x^\alpha\right). \tag{38}$$

We obtain

$$\left(E_\alpha\left(\frac{1}{a}x^\alpha\right)-p_\alpha(x^\alpha)\right)^{\otimes 2}=1+\left(p_\alpha(x^\alpha)\right)^{\otimes 2}. \tag{39}$$

So,

$$\left(E_\alpha\left(\frac{1}{a}x^\alpha\right)\right)^{\otimes 2}-1=2E_\alpha\left(\frac{1}{a}x^\alpha\right)\otimes p_\alpha(x^\alpha). \tag{40}$$

Hence,

$$p_\alpha(x^\alpha)=\frac{1}{2}\left(E_\alpha\left(\frac{1}{a}x^\alpha\right)-E_\alpha\left(-\frac{1}{a}x^\alpha\right)\right). \tag{41}$$

That is,

$$\left({}_0D_x^\alpha\right)[y_\alpha(x^\alpha)]=\frac{1}{2}\left(E_\alpha\left(\frac{1}{a}x^\alpha\right)-E_\alpha\left(-\frac{1}{a}x^\alpha\right)\right). \tag{42}$$

Therefore,

$$\begin{aligned} y_\alpha(x^\alpha)&=\left({}_0I_x^\alpha\right)\left[\frac{1}{2}\left(E_\alpha\left(\frac{1}{a}x^\alpha\right)-E_\alpha\left(-\frac{1}{a}x^\alpha\right)\right)\right]\\ &=\frac{a}{2}\left(E_\alpha\left(\frac{1}{a}x^\alpha\right)+E_\alpha\left(-\frac{1}{a}x^\alpha\right)\right)+D. \end{aligned} \tag{43}$$

Since $y_\alpha(0)=a$, it follows that $D=0$, and hence

$$y_\alpha(x^\alpha)=\frac{a}{2}\left(E_\alpha\left(\frac{1}{a}x^\alpha\right)+E_\alpha\left(-\frac{1}{a}x^\alpha\right)\right).$$

IV. CONCLUSION

In this paper, we find the fractional integral curves of some fractional differential equations based on Jumarie type of R-L fractional calculus. A new multiplication of fractional analytic functions and product rule for fractional derivatives play important roles in this paper. In fact, our results are generalizations of the results in ordinary differential equations. In the future, we will continue to use Jumarie's modified R-L fractional calculus and the new multiplication of fractional analytic functions to study the problems in fractional differential equations and applied mathematics.

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